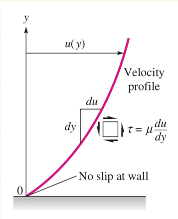
工程流体力学

I

Chapter 1

1. Newtonian fluid (牛顿流体): A fluid which has a linear relationship (直线关系) between shear stress (剪应力) and velocity gradient (速度梯度).



$$\boxed{\tau = \mu \frac{du}{dy}}$$

μ: 粘度系数. 单位: kg/m·s
τ: 剪应力(某一点)
u: 速度, y: 间隔高度.

2. boundary layer: Flows constrained by solid surface
   a. Flow near a bounding surface with:
      1. significant velocity gradients
      2. significant shear stresses.
   b. Flows far from bounding surface with:
      1. negligible velocity gradients
      2. negligible shear stresses
      3. significant inertia effects (惯性效力).

3. Reynolds number: $\nu = \frac{\mu}{\rho}$

$$\boxed{Re = \frac{\rho V L}{\mu} = \frac{VL}{\nu}}$$

无量纲. V: 速度. L: 特征尺度.

4. Streamline 流线: a line everywhere tangent (相切) to the velocity vector at a given instant.

$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dr}{V}}$$

$\xRightarrow{积分}$ $x = \int u\,dt$, $y = \int v\,dt$, $z = \int w\,dt$

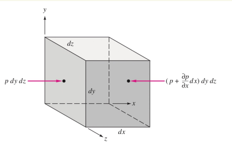
流体中的三种力: 黏性力、重力、压力.
[微尺度(nm, um): 表面张力]

Secondary dimension	SI unit	BG unit	Conversion factor
Area $\{L^2\}$	m^2	ft^2	$1\ m^2 = 10.764\ ft^2$
Volume $\{L^3\}$	m^3	ft^3	$1\ m^3 = 35.315\ ft^3$
Velocity $\{LT^{-1}\}$	m/s	ft/s	1 ft/s = 0.3048 m/s
Acceleration $\{LT^{-2}\}$	m/s^2	ft/s^2	$1\ ft/s^2 = 0.3048\ m/s^2$
Pressure or stress $\{ML^{-1}T^{-2}\}$	$Pa = N/m^2$	lbf/ft^2	$1\ lbf/ft^2 = 47.88\ Pa$
Angular velocity $\{T^{-1}\}$	s^{-1}	s^{-1}	$1\ s^{-1} = 1\ s^{-1}$
Energy, heat, work $\{ML^2T^{-2}\}$	$J = N \cdot m$	$ft \cdot lbf$	$1\ ft \cdot lbf = 1.3558\ J$
Power $\{ML^2T^{-3}\}$	W = J/s	$ft \cdot lbf/s$	$1\ ft \cdot lbf/s = 1.3558\ W$
Density $\{ML^{-3}\}$	kg/m^3	$slugs/ft^3$	$1\ slug/ft^3 = 515.4\ kg/m^3$
Viscosity $\{ML^{-1}T^{-1}\}$	$kg/(m \cdot s)$	$slugs/(ft \cdot s)$	$1\ slug/(ft \cdot s) = 47.88\ kg/(m \cdot s)$
Specific heat $\{L^2T^{-2}\Theta^{-1}\}$	$m^2/(s^2 \cdot K)$	$ft^2/(s^2 \cdot °R)$	$1\ m^2/(s^2 \cdot K) = 5.980\ ft^2/(s^2 \cdot °R)$

Mach number 马赫数： $Ma = \frac{V}{a}$ V：速度，a：声速 [$a_{ideal\ gas} = (kRT)^{\frac{1}{2}} \approx 343 m/s$]

Chapter 2

1. Pressure Force on a Fluid Element



x方向: $dF_x = p\,dy\,dz - (p+\frac{\partial p}{\partial x}dx)\,dy\,dz = -\frac{\partial p}{\partial x}\,dx\,dy\,dz$.

$\Longrightarrow d\vec{F} = (-\vec{i}\frac{\partial p}{\partial x} - \vec{j}\frac{\partial p}{\partial y} - \vec{k}\frac{\partial p}{\partial z})\,dx\,dy\,dz$

$\Longrightarrow \vec{f}_{press} = -\nabla p$ f: 单位体积流体的净力.

2. Equilibrium of a Fluid Element.

仅考虑重力: $d\vec{F} = \rho\vec{g}\,dx\,dy\,dz \iff \vec{f}_{grav} = \rho\vec{g}$, $\vec{g}$: 重力加速度

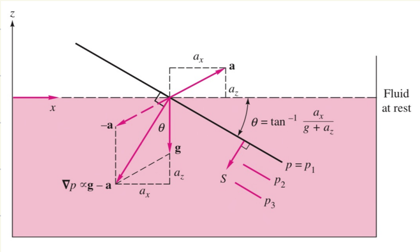
对于不可压缩恒定粘度流体: $\vec{f}_{vs} = \mu\left(\frac{\partial^2\vec{V}}{\partial x^2} + \frac{\partial^2\vec{V}}{\partial y^2} + \frac{\partial^2\vec{V}}{\partial z^2}\right) = \mu\nabla^2\vec{V}$, μ: 粘度系数, $\vec{V}$: 速度矢量

牛顿第二定律(仅压力、重力、粘性力): $\rho\vec{a} = \sum\vec{f} = \vec{f}_{press} + \vec{f}_{grav} + \vec{f}_{visc} = -\nabla p + \rho\vec{g} + \mu\nabla^2\vec{V}$.

$$\iff \boxed{\nabla p = \rho(\vec{g} - \vec{a}) + \mu\nabla^2\vec{V}}$$

梯度: $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

3. Pressure Distribution in Rigid-Body Motion.



$$\boxed{\theta = \tan^{-1}\frac{a_x}{g + a_z}}$$

压力增加率: $\boxed{\frac{dp}{ds} = \rho G}$ $G = [a_x^2 + (g + a_z)^2]^{\frac{1}{2}}$

(无关容器)

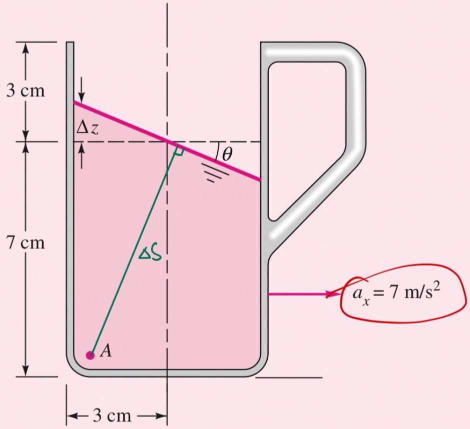
G看作等效重力加速度.

EXAMPLE 2.13

A drag racer rests her coffee mug on a horizontal tray while she accelerates at 7 m/s^2. The mug is 10 cm deep and 6 cm in diameter and contains coffee 7 cm deep at rest. (*a*) Assuming rigid-body acceleration of the coffee, determine whether it will spill out of the mug. (*b*) Calculate the gage (表压) pressure in the corner at point *A* if the density of coffee is 1010 kg/m^3.

Solution

- *System sketch:* Figure E2.13 shows the coffee tilted during the acceleration.



E2.13

对称的

- *Assumptions:* Rigid-body horizontal acceleration, $a_x = 7$ m/s^2. Symmetric coffee cup.
- *Property values:* Density of coffee given as 1010 kg/m^3.
- *Approach (a):* Determine the angle of tilt from the known acceleration, then find the height rise.
- *Solution steps:* From Eq. (2.39), the angle of tilt is given by

$$\boxed{\theta = \tan^{-1}\frac{a_x}{g}} = \tan^{-1}\frac{7.0\ \text{m/s}^2}{9.81\ \text{m/s}^2} = 35.5°$$

If the mug is symmetric, the tilted surface will pass through the center point of the rest position, as shown in Fig. E2.13. Then the rear side of the coffee free surface will rise an amount Δz given by

$$\Delta z = (3\ \text{cm})(\tan 35.5°) = 2.14\ \text{cm} < 3\ \text{cm} \qquad \text{therefore no spilling} \qquad \textit{Ans. (a)}$$

- *Comment (a):* This solution neglects sloshing, which might occur if the start-up is uneven.

垂直的

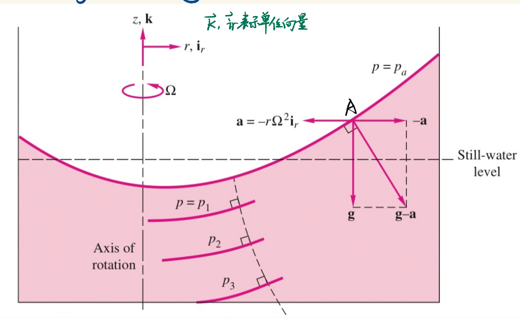
- *Approach (b):* The pressure at A can be computed from Eq. (2.40), using the perpendicular distance Δs from the surface to A. When at rest, $p_A = \rho g h_{\text{rest}} = (1010\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.07\ \text{m}) = 694$ Pa. When accelerating,

$$p_A = \rho G\,\Delta s = \left(1010\frac{\text{kg}}{\text{m}^3}\right)\left[\sqrt{(9.81)^2 + (7.0)^2}\right][(0.07 + 0.0214)\cos 35.5°] \approx 906\ \text{Pa} \quad \textit{Ans. (b)}$$

- *Comment (b):* The acceleration has increased the pressure at A by 31 percent. Think about this alternative: why does it work? Since $a_z = 0$, we may proceed vertically down the left side to compute

$$p_A = \rho g(z_{\text{surf}} - z_A) = (1010\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.0214 + 0.07\ \text{m}) = 906\ \text{Pa}$$

4. Rigid-Body Rotation.



A点：角速度：$\vec{\Omega}=\vec{k}\Omega$　　位置矢量：$\vec{r_0}=\vec{i_r}r$

$\Rightarrow$ 加速度：$\vec{a}=\vec{\Omega}\times(\vec{\Omega}\times\vec{r_0})=-r\Omega^2\vec{i_r}$

由 $\vec{\nabla}p=\rho(\vec{g}-\vec{a})+\mu\nabla^2\vec{V}$，而整体转动，无速度梯度.

$$\vec{\nabla}p=\vec{i_r}\frac{\partial p}{\partial r}+\vec{k}\frac{\partial p}{\partial z}=\rho(\vec{g}-\vec{a})=\rho(-g\vec{k}+r\Omega^2\vec{i_r})$$

比较得：$\frac{\partial p}{\partial r}=\rho r\Omega^2$　　$\frac{\partial p}{\partial z}=-\rho g=-\gamma$

偏微分方程求解：$p=\frac{1}{2}\rho r^2\Omega^2+f(z)$

$$\Rightarrow \frac{\partial p}{\partial z}=0+f'(z)=-\gamma \Rightarrow f(z)=-\gamma z+C \quad 代入：p=C-\gamma z+\frac{1}{2}\rho r^2\Omega^2$$

$$\boxed{p=p_0-\gamma z+\frac{1}{2}\rho r^2\Omega^2},\ \gamma=\rho g$$

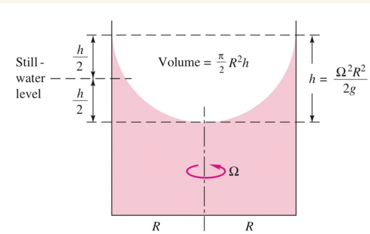


Fig. 2.23

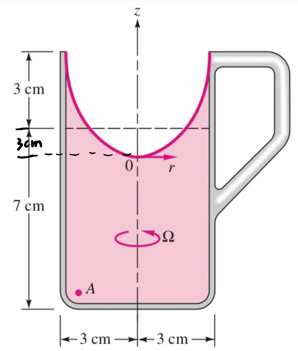
$$z=\frac{p_0-p_1}{\gamma}+\frac{r^2\Omega^2}{2g}=a+br^2$$

抛物面.

最高点：$p=p_0$，$h=\frac{\Omega^2R^2}{2g}$.

EXAMPLE 2.14

The coffee cup in Example 2.13 is removed from the drag racer, placed on a turntable, and 转动 rotated about its central axis until a rigid-body mode occurs. Find (*a*) the 角速度 angular velocity that will cause the coffee to just reach the lip of the cup and (*b*) the gage pressure at point *A* for this condition.



Solution

The cup contains 7 cm of coffee. The remaining distance of 3 cm up to the lip must equal the distance $h/2$ in Fig. 2.23. Thus

a)
$$\frac{h}{2} = 0.03 \text{ m} = \frac{\Omega^2 R^2}{4g} = \frac{\Omega^2 (0.03 \text{ m})^2}{4(9.81 \text{ m/s}^2)}$$

Solving, we obtain

$$\Omega^2 = 1308 \qquad \text{or} \qquad \Omega = 36.2 \text{ rad/s} = 345 \text{ r/min} \qquad \textit{Ans. (a)}$$

To compute the pressure, it is convenient to put the origin of coordinates r and z at the bottom of the free-surface depression, as shown in Fig. E2.14. The gage pressure here is $p_0 = 0$, and point A is at $(r, z) = (3 \text{ cm}, -4 \text{ cm})$. Equation (2.46) can then be evaluated:

b)
$$\begin{aligned} p_A &= 0 - (1010 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(-0.04 \text{ m}) \\ &\quad + \tfrac{1}{2}(1010 \text{ kg/m}^3)(0.03 \text{ m})^2(1308 \text{ rad}^2/\text{s}^2) \\ &= 396 \text{ N/m}^2 + 594 \text{ N/m}^2 = 990 \text{ Pa} \end{aligned} \qquad \textit{Ans. (b)}$$

This is about 43 percent greater than the still-water pressure $p_A = 694$ Pa.

Chapter 3 积分

三大公理：1. 质量守恒：$m_{sys} = \int_{sys} \rho dv$, $\frac{dm}{dt}\Big|_{sys} = 0$

2. 动量：$\vec{P}_{sys} = m_{sys}\vec{V} = \int_{sys} \vec{V} \rho dv$, $\sum \vec{F} = \frac{d\vec{P}}{dt}\Big|_{sys} = \frac{dm\vec{V}}{dt}\Big|_{sys}$

3. 能量：$E_{sys} = \int_{sys} e \rho dv$ e: total energy per unit mass (includes kinetic 动能, potential 势能 and internal 内能 energy).

1. One-Dimension Fixed Control Volume.

B: any property (属性) of the fluid (energy, momentum 动量 ...)

$\beta \triangleq \frac{dB}{dm}$, $B_{cv} = \int_{cv} \beta dm = \int_{cv} \beta \rho dV$

$$\Rightarrow \frac{d}{dt}(B_{cv}) = \frac{1}{dt} B_{cv}(t+dt) - \frac{1}{dt} B_{cv}(t) = \frac{1}{dt}[B_2(t+dt) - (\beta\rho dV)_{out} + (\beta\rho dV)_{in}] - \frac{1}{dt}[B_2(t)]$$

$$= \frac{1}{dt}[B_2(t+dt) - B_2(t)] - (\beta\rho dV)_{out} + (\beta\rho dV)_{in}$$

one-dimensional Reynolds transport theorem for a fixed volume:

$$\frac{d}{dt}(B_{sys}) = \frac{d}{dt}\left(\int_{cv} \beta\rho dv\right) + (\beta\rho dV)_{out} - (\beta\rho dV)_{in}.$$

2. Reynolds Transport Theorem (control volume: c.v., V: 速度, A: 面积)

$$\boxed{\frac{dB}{dt}\Big|_{sys} = \frac{\partial}{\partial t}\int_{cv} \beta\rho dV + \int_{A_e} \beta_e \rho_e V_e dA_e - \int_{A_i} \beta_i \rho_i V_i dA_i}$$

↓ System rate of change of B → transient term

↓ Rate of change of B in c.v.

↓ Rate of B leaving c.v.

↓ Rate of B entering c.v.

→ convective terms

3. Conservation of Mass: $\left(\dot{m} = \int \rho V dA = \rho A V = \rho Q \right.$, A:面积, V:速度)

$B = m$, $\beta = 1$, $\frac{dB}{dt} = 0$

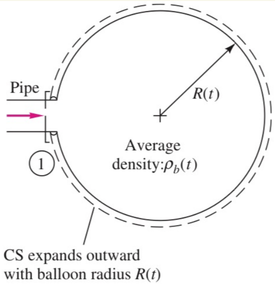
Reynolds Transport Theorem becomes:

$$\boxed{\left.\frac{dm}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{cv} \rho dV + \int_{A_e} \rho_e \vec{V_e} \cdot d\vec{A_e} - \int_{A_i} \rho_i \vec{V_i} \cdot d\vec{A_i} = 0}$$

Rate of change of mass in c.v.	Rate of mass leaving c.v.	Rate of mass entering c.v.
$=0$ for steady-state	$= \dot{m}_e$	$= \dot{m}_i$

$$\Longrightarrow \left.\frac{dm}{dt}\right|_{cv} + \sum \dot{m}_e - \sum \dot{m}_i = 0$$

$\Longrightarrow$ steady-state: $\sum \dot{m}_e = \sum \dot{m}_i$



E3.2

EXAMPLE 3.2

The balloon in Fig. E3.2 is being filled through section 1, where the area is A_1, velocity is V_1, and fluid density is ρ_1. The average density within the balloon is $\rho_b(t)$. Find an expression for the rate of change of system mass within the balloon at this instant.

Solution

- *System sketch:* Figure E3.2 shows one inlet (入), no exits (出). The control volume and system expand together, hence the relative velocity $V_r = 0$ on the balloon surface.
- *Assumptions:* Unsteady flow (the control volume mass increases), deformable (可变形) control surface, one-dimensional inlet conditions.
- *Approach:* Apply Eq. (3.16) with $V_r = 0$ on the balloon surface and $V_r = V_1$ at the inlet.
- *Solution steps:* The property being studied is *mass*, $B = m$ and $\beta = dm/dm =$ unity. Apply Eq. (3.16). The volume integral is evaluated based on average density ρ_b, and the surface integral term is negative (for an inlet):

$$\left(\frac{dm}{dt}\right)_{syst} = \frac{d}{dt}\left(\int_{CV} \rho\, d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V}_r \cdot \mathbf{n})dA = \frac{d}{dt}\left(\rho_b \frac{4\pi}{3} R^3\right) - \rho_1 A_1 V_1 \qquad \textit{Ans.}$$

- *Comments:* The relation given is the answer to the question that was asked. Actually, by the conservation law for mass, Eq. (3.1), $(dm/dt)_{syst} = 0$, and the answer could be rewritten as

$$\frac{d}{dt}(\rho_b R^3) = \frac{3}{4\pi}\rho_1 A_1 V_1$$

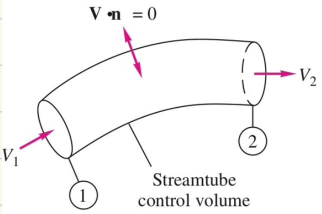
This is a first-order ordinary differential equation relating gas density and balloon radius. It could form part of an engineering analysis of balloon inflation. It cannot be solved without further use of mechanics and thermodynamics to relate the four unknowns ρ_b, ρ_1, V_1, and R. The pressure and temperature and the elastic properties of the balloon would also have to be brought into the analysis.

Incompressible Flow:

$$\sum Q_{in} = \sum Q_{out} \qquad Q = VA$$

$$Q_{cs} = \int (\vec{V}\cdot\vec{n})\,dA$$

volume flow: $\boxed{V_{av} = \frac{Q}{A} = \frac{1}{A}\int (\vec{V}\cdot\vec{n})\,dA}$



E3.3

EXAMPLE 3.3

Write the conservation-of-mass relation for steady flow through a streamtube (flow everywhere parallel to the walls) with a single one-dimensional inlet 1 and exit 2 (Fig. E3.3).

Solution

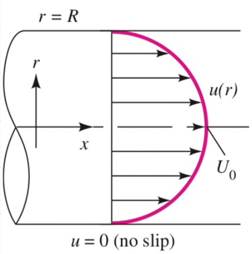
For steady flow Eq. (3.24) applies with the single inlet and exit:

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \text{const}$$

Thus, in a streamtube in steady flow, the mass flow is constant across every section of the tube. If the density is constant, then

$$Q = A_1 V_1 = A_2 V_2 = \text{const} \qquad \text{or} \qquad V_2 = \frac{A_1}{A_2} V_1$$

The volume flow is constant in the tube in steady incompressible flow, and the velocity increases as the section area decreases. This relation was derived by Leonardo da Vinci in 1500.



E3.4

EXAMPLE 3.4

速度分布

For steady viscous flow through a circular tube (Fig. E3.4), the axial velocity profile is given approximately by

$$u = U_0\left(1 - \frac{r}{R}\right)^m$$

so that u varies from zero at the wall ($r = R$), or no slip, up to a maximum $u = U_0$ at the centerline $r = 0$. For highly viscous (laminar 层流) flow $m \approx \frac{1}{2}$, while for less viscous (turbulent 湍流) flow $m \approx \frac{1}{7}$. Compute the average velocity if the density is constant.

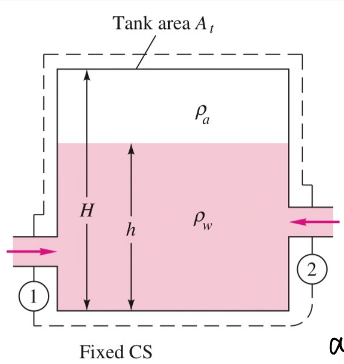
Solution

The average velocity is defined by Eq. (3.32). Here $\mathbf{V} = \mathbf{i}u$ and $\mathbf{n} = \mathbf{i}$, and thus $\mathbf{V}\cdot\mathbf{n} = u$. Since the flow is symmetric, the differential area can be taken as a circular strip $dA = 2\,\pi r\,dr$. Equation (3.32) becomes

$$V_{av} = \frac{1}{A}\int u\,dA = \frac{1}{\pi R^2}\int_0^R U_0\left(1 - \frac{r}{R}\right)^m 2\pi r\,dr$$

or

$$V_{av} = U_0 \frac{2}{(1+m)(2+m)} \qquad \textit{Ans.}$$



EXAMPLE 3.5

入口

The tank in Fig. E3.5 is being filled with water by two one-dimensional inlets. Air is trapped at the top of the tank. The water height is h. (*a*) Find an expression for the change in water height dh/dt. (*b*) Compute dh/dt if $D_1 = 1$ in, $D_2 = 3$ in, $V_1 = 3$ ft/s, $V_2 = 2$ ft/s, and $A_t = 2\ \text{ft}^2$, assuming water at 20°C.

Solution

A suggested control volume encircles the tank and cuts through the two inlets. The flow within is unsteady, and Eq. (3.22) applies with no outlets and two inlets:

a).

$$\frac{d}{dt}\left(\int_{CV} \rho\, d\mathcal{V}\right) - \rho_1 A_1 V_1 - \rho_2 A_2 V_2 = 0 \tag{1}$$

Now if A_t is the tank cross-sectional area, the unsteady term can be evaluated as follows:

0 质量不变.

$$\frac{d}{dt}\left(\int_{CV} \rho\, d\mathcal{V}\right) = \frac{d}{dt}(\rho_w A_t h) + \frac{d}{dt}[\rho_a A_t (H - h)] = \rho_w A_t \frac{dh}{dt} \tag{2}$$

The ρ_a term vanishes because it is the rate of change of air mass and is zero because the air is trapped at the top. Substituting (2) into (1), we find the change of water height

$$\frac{dh}{dt} = \frac{\rho_1 A_1 V_1 + \rho_2 A_2 V_2}{\rho_w A_t} \qquad \textit{Ans. (a)}$$

For water, $\rho_1 = \rho_2 = \rho_w$, and this result reduces to

$$\frac{dh}{dt} = \frac{A_1 V_1 + A_2 V_2}{A_t} = \frac{Q_1 + Q_2}{A_t} \tag{3}$$

b). The two inlet volume flows are

$$Q_1 = A_1 V_1 = \tfrac{1}{4}\pi(\tfrac{1}{12}\ \text{ft})^2(3\ \text{ft/s}) = 0.016\ \text{ft}^3/\text{s}$$

$$Q_2 = A_2 V_2 = \tfrac{1}{4}\pi(\tfrac{3}{12}\ \text{ft})^2(2\ \text{ft/s}) = 0.098\ \text{ft}^3/\text{s}$$

Then, from Eq. (3),

$$\frac{dh}{dt} = \frac{(0.016 + 0.098)\ \text{ft}^3/\text{s}}{2\ \text{ft}^2} = 0.057\ \text{ft/s} \qquad \textit{Ans. (b)}$$

Suggestion: Repeat this problem with the top of the tank open.

4. Linear Momentum

$$B = mV \ , \quad \frac{dB}{dt} = \frac{dmV}{dt} = ma = F \ , \quad \beta = V$$

Reynolds Transport Theorem becomes:

$$\boxed{\sum \vec{F} = \frac{dm\vec{V}}{dt}\bigg|_{sys} = \frac{\partial}{\partial t}\int_{cv} \vec{V}\rho \, dv + \int_{A_e} \vec{V}\rho_e \vec{V_e}\cdot d\vec{A_e} - \int_{A_i} \vec{V}\rho_i \vec{V_i}\cdot d\vec{A_i}}$$

↓ the Σ of the external forces acting on the c.v.

↓ the rate of change of momentum in the c.v.

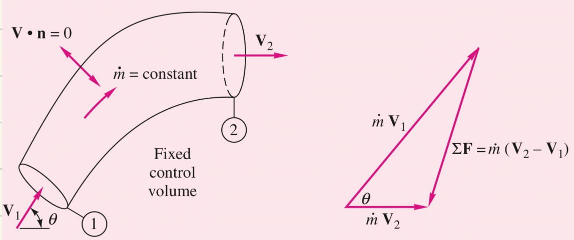
↓ the rate of momentum leaving the c.v.

↓ the rate of momentum entering the c.v.

$\Rightarrow$ steady-state : $\sum \vec{F} = \dot{m}_e \vec{V} - \dot{m}_i \vec{V}$

EXAMPLE 3.7

A fixed control volume of a streamtube in steady flow has a 统一的 uniform inlet flow (ρ_1, A_1, V_1) and a uniform exit flow (ρ_2, A_2, V_2), as shown in Fig. 3.7. Find an expression for the net force on the control volume.



(*a*) (*b*)

Solution

Equation (3.40) applies with one inlet and exit:

$$\sum \mathbf{F} = \dot{m}_2\mathbf{V}_2 - \dot{m}_1\mathbf{V}_1 = (\rho_2 A_2 V_2)\mathbf{V}_2 - (\rho_1 A_1 V_1)\mathbf{V}_1$$

The volume integral term vanishes for steady flow, but from conservation of mass in Example 3.3 we saw that

$$\dot{m}_1 = \dot{m}_2 = \dot{m} = \text{const}$$

Therefore a simple form for the desired result is

$$\sum \mathbf{F} = \dot{m}(\mathbf{V}_2 - \mathbf{V}_1) \qquad \textit{Ans.}$$

This is a *vector* relation and is sketched in Fig. 3.7*b*. The term Σ **F** represents the net force acting on the control volume due to all causes; it is needed to balance the change in momentum of the fluid as it turns and decelerates while passing through the control volume.

EXAMPLE 3.9

A water jet of velocity V_j impinges normal to a flat plate that moves to the right at velocity V_c, as shown in Fig. 3.9*a*. Find the force required to keep the plate moving at constant velocity if the jet density is 1000 kg/m^3, the jet area is 3 cm^2, and V_j and V_c are 20 and 15 m/s, respectively. 忽略 Neglect the weight of the jet and plate, and assume steady flow with respect to the moving plate with the jet splitting into an equal upward and downward half-jet.

Solution

The suggested control volume in Fig. 3.9*a* cuts through the plate support to expose the desired forces R_x and R_y. This control volume moves at speed V_c and thus is fixed relative to the plate, as in Fig. 3.9*b*. We must satisfy both mass and momentum conservation for the assumed steady flow pattern in Fig. 3.9*b*. There are two outlets and one inlet, and Eq. (3.30) applies for mass conservation:

$$\dot{m}_{\text{out}} = \dot{m}_{\text{in}}$$

or

$$\rho_1 A_1 V_1 + \rho_2 A_2 V_2 = \rho_j A_j (V_j - V_c) \tag{1}$$

We assume that the water is incompressible $\rho_1 = \rho_2 = \rho_j$, and we are given that $A_1 = A_2 = \frac{1}{2}A_j$. Therefore Eq. (1) reduces to

$$V_1 + V_2 = 2(V_j - V_c) \tag{2}$$

对称

Strictly speaking, this is all that mass conservation tells us. However, from the symmetry of the jet deflection and the neglect of gravity on the fluid trajectory, we conclude that the two velocities V_1 and V_2 must be equal, and hence Eq. (2) becomes

$$V_1 = V_2 = V_j - V_c \tag{3}$$

This equality can also be predicted by Bernoulli's equation in Sect 3.5. For the given numerical values, we have

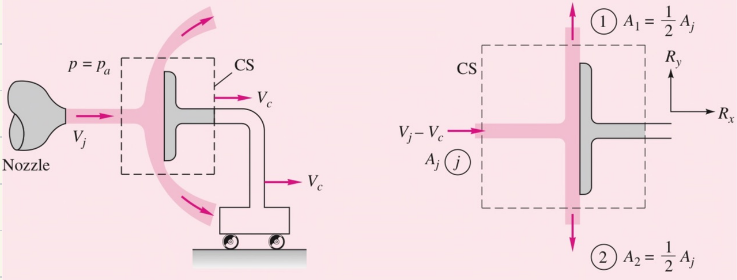
$$V_1 = V_2 = 20 - 15 = 5 \text{ m/s}$$

Now we can compute R_x and R_y from the two components of momentum conservation. Equation (3.40) applies with the unsteady term zero:

$$\sum F_x = R_x = \dot{m}_1 u_1 + \dot{m}_2 u_2 - \dot{m}_j u_j \tag{4}$$

where from the mass analysis, $\dot{m}_1 = \dot{m}_2 = \frac{1}{2}\dot{m}_j = \frac{1}{2}\rho_j A_j (V_j - V_c)$. Now check the flow directions at each section: $u_1 = u_2 = 0$, and $u_j = V_j - V_c = 5$ m/s. Thus Eq. (4) becomes

$$R_x = -\dot{m}_j u_j = -[\rho_j A_j (V_j - V_c)](V_j - V_c) \tag{5}$$



(*a*) (*b*)

For the given numerical values we have

$$R_x = -(1000\ \text{kg/m}^3)(0.0003\ \text{m}^2)(5\ \text{m/s})^2 = -7.5\ (\text{kg}\cdot\text{m})/\text{s}^2 = -7.5\ \text{N} \qquad \textit{Ans.}$$

This acts to the *left*; that is, it requires a restraining force to keep the plate from accelerating to the right due to the continuous impact of the jet. The vertical force is

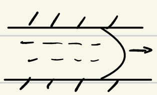
$$F_y = R_y = \dot{m}_1 v_1 + \dot{m}_2 v_2 - \dot{m}_j v_j$$

Check directions again: $v_1 = V_1$, $v_2 = -V_2$, $v_j = 0$. Thus

$$R_y = \dot{m}_1(V_1) + \dot{m}_2(-V_2) = \tfrac{1}{2}\dot{m}_j(V_1 - V_2) \qquad (6)$$

But since we found earlier that $V_1 = V_2$, this means that $R_y = 0$, as we could expect from the symmetry of the jet deflection.[9] Two other results are of interest. First, the relative velocity at section 1 was found to be 5 m/s up, from Eq. (3). If we convert this to absolute motion by adding on the control-volume speed $V_c = 15$ m/s to the right, we find that the absolute velocity $\mathbf{V}_1 = 15\mathbf{i} + 5\mathbf{j}$ m/s, or 15.8 m/s at an angle of 18.4° upward, as indicated in Fig. 3.9*a*. Thus the absolute jet speed changes after hitting the plate. Second, the computed force R_x does not change if we assume the jet deflects in all radial directions along the plate surface rather than just up and down. Since the plate is normal to the x axis, there would still be zero outlet x-momentum flux when Eq. (4) was rewritten for a radial deflection condition.

5、Momentum Flux Correction Factor



$$\rho\int u^2 dA = \beta \dot{m} V_{av} = \beta\rho A V_{av}^2$$

The turbulent correction factors have the following range of values:

Turbulent flow:	m	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$
	β	1.037	1.027	1.020	1.016	1.013

6. Energy Equation

$B = E = \int_{cv} e\rho dV$, $\beta = e = u + \frac{1}{2}V^2 + gz$

Reynolds Transport Theorem becomes:

$$\dot{Q} - \dot{W} = \frac{dE}{dt}\bigg|_{sys} = \frac{\partial}{\partial t}\int_{cv} e\rho dV + \int_{cv} e_e\rho_e \vec{V}_e d\vec{A}_e - \int_{cv} e_i\rho_i \vec{V}_i d\vec{A}_i$$

$$\Rightarrow \boxed{\left(\frac{P}{\rho g} + \frac{\alpha}{2g}V^2 + z\right)_{in} = \left(\frac{P}{\rho g} + \frac{\alpha}{2g}V^2 + z\right)_{out} + h_{turbine} - h_{pump} + h_{friction}}$$, h：高度.

For pumps: $h_p = \frac{w_s}{g}$, w_s: the useful work per unit mass to the fluid.

$\Rightarrow w_s = g h_p \Rightarrow P = \dot{W}_f = \dot{m} w_s = (\rho Q)(g h_p)$.

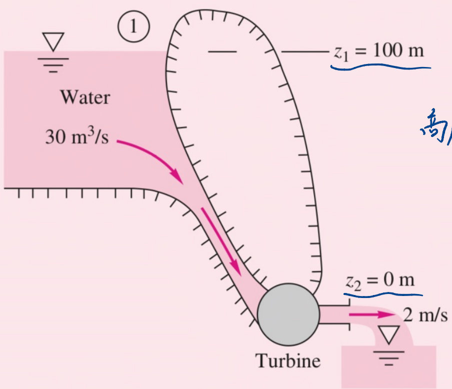
EXAMPLE 3.23

水力发电的

A hydroelectric power plant (Fig. E3.23) takes in 30 m^3/s of water through its turbine and discharges it to the atmosphere at $V_2 = 2$ m/s. The head loss in the turbine and penstock system is $h_f = 20$ m. Assuming turbulent flow, $\alpha \approx 1.06$, estimate the power in MW extracted by the turbine.

Solution

We neglect viscous work and heat transfer and take section 1 at the reservoir surface (Fig. E3.23), where $V_1 \approx 0$, $p_1 = p_{\text{atm}}$, and $z_1 = 100$ m. Section 2 is at the turbine outlet.



高度

E3.23

The steady flow energy equation (3.75) becomes, in head form,

$$\frac{p_1}{\gamma} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{\alpha_2 V_2^2}{2g} + z_2 + h_t + h_f$$

$$\frac{p_a}{\gamma} + \frac{1.06(0)^2}{2(9.81)} + 100 \text{ m} = \frac{p_a}{\gamma} + \frac{1.06(2.0 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} + 0 \text{ m} + h_t + 20 \text{ m}$$

The pressure terms cancel, and we may solve for the turbine head (which is positive):

$$h_t = 100 - 20 - 0.2 \approx 79.8 \text{ m}$$

The turbine extracts about 79.8 percent of the 100-m head available from the dam. The total power extracted may be evaluated from the water mass flow:

$$P = \dot{m}w_s = (\rho Q)(gh_t) = (998 \text{ kg/m}^3)(30 \text{ m}^3\text{/s})(9.81 \text{ m/s}^2)(79.8 \text{ m})$$

$$= 23.4 \text{ E6 kg} \cdot \text{m}^2\text{/s}^3 = 23.4 \text{ E6 N} \cdot \text{m/s} = 23.4 \text{ MW} \qquad \textit{Ans.}$$

The turbine drives an electric generator that probably has losses of about 15 percent, so the net power generated by this hydroelectric plant is about 20 MW.

7. The Bernoulli Equation.

$$\boxed{\frac{P_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{P_2}{\rho}+\frac{V_2^2}{2}+gz_2=\text{const.}}$$

非黏性流体.

for ① steady, ② incompressible, ③ frictionless flow ④ between two points along a stream line.

由动量方程推导: 🔗P170

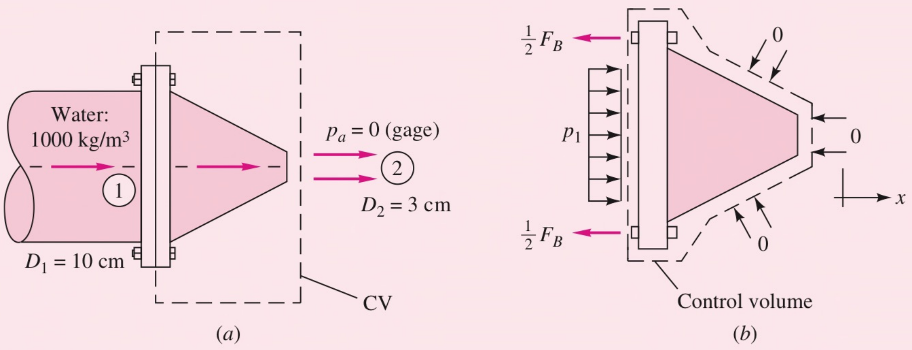
$$\int_1^2 \frac{\partial V}{\partial t}ds+\int_1^2\frac{dp}{\rho}+\frac{1}{2}(V_2^2-V_1^2)+g(z_2-z_1)=0.$$

EXAMPLE 3.16

A 10-cm fire hose with a 3-cm nozzle discharges 1.5 m^3/min to the atmosphere. Assuming frictionless flow, find the force F_B exerted by the flange bolts to hold the nozzle on the hose.

Solution

We use Bernoulli's equation and continuity to find the pressure p_1 upstream of the nozzle, and then we use a control volume momentum analysis to compute the bolt force, as in Fig. E3.16.



E3.16

The flow from 1 to 2 is a constriction exactly similar in effect to the venturi in Example 3.15, for which Eq. (1) gave

$$p_1 = p_2 + \tfrac{1}{2}\rho(V_2^2 - V_1^2) \quad \text{伯努利方程} \tag{1}$$

The velocities are found from the known flow rate $Q = 1.5\ \mathrm{m^3/min}$ or $0.025\ \mathrm{m^3/s}$:

$$V_2 = \frac{Q}{A_2} = \frac{0.025\ \mathrm{m^3/s}}{(\pi/4)(0.03\ \mathrm{m})^2} = 35.4\ \mathrm{m/s}$$

$$V_1 = \frac{Q}{A_1} = \frac{0.025\ \mathrm{m^3/s}}{(\pi/4)(0.1\ \mathrm{m})^2} = 3.2\ \mathrm{m/s}$$

We are given $p_2 = p_a = 0$ gage pressure. Then Eq. (1) becomes

$$p_1 = \tfrac{1}{2}(1000\ \mathrm{kg/m^3})[(35.4^2 - 3.2^2)\mathrm{m^2/s^2}]$$
$$= 620{,}000\ \mathrm{kg/(m \cdot s^2)} = 620{,}000\ \mathrm{Pa\ gage}$$

The control volume force balance is shown in Fig. E3.16*b*:

$$\sum F_x = -F_B + p_1A_1$$

and the zero gage pressure on all other surfaces contributes no force. The x-momentum flux is $+\dot{m}V_2$ at the outlet and $-\dot{m}V_1$ at the inlet. The steady flow momentum relation (3.40) thus gives

$$-F_B + p_1A_1 = \dot{m}(V_2 - V_1) \quad \text{动量方程}$$

or

$$F_B = p_1A_1 - \dot{m}(V_2 - V_1) \tag{2}$$

Substituting the given numerical values, we find

$$\dot{m} = \rho Q = (1000\ \mathrm{kg/m^3})(0.025\ \mathrm{m^3/s}) = 25\ \mathrm{kg/s}$$

$$A_1 = \frac{\pi}{4}D_1^2 = \frac{\pi}{4}(0.1\ \mathrm{m})^2 = 0.00785\ \mathrm{m^2}$$

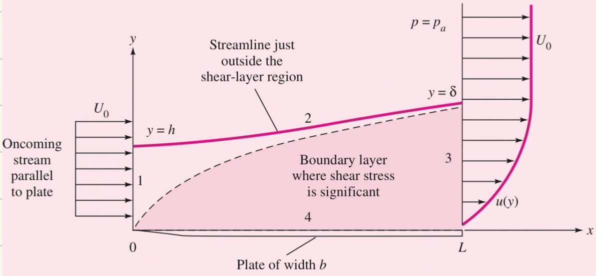
$$F_B = (620{,}000\ \mathrm{N/m^2})(0.00785\ \mathrm{m^2}) - (25\ \mathrm{kg/s})[(35.4 - 3.2)\mathrm{m/s}]$$
$$= 4872\ \mathrm{N} - 805\ \mathrm{(kg \cdot m)/s^2} = 4067\ \mathrm{N}\ (915\ \mathrm{lbf}) \qquad \textit{Ans.}$$

EXAMPLE 3.11

Example 3.9 treated a plate at normal incidence to an oncoming flow. In Fig. 3.10 the plate is parallel to the flow. The stream is not a jet but a broad river, or *free stream,* of uniform velocity $\mathbf{V} = U_0\mathbf{i}$. The pressure is assumed uniform, and so it has no net force on the plate. The plate does not block the flow as in Fig. 3.9, so the only effect is due to boundary shear, which was neglected in the previous example. The no-slip condition at the wall brings the fluid there to a halt, and these slowly moving particles retard their neighbors above, so that at the end of the plate there is a significant retarded shear layer, or *boundary layer,* of thickness $y = \delta$. The viscous stresses along the wall can sum to a finite drag force on the plate. These effects are illustrated in Fig. 3.10. The problem is to make an integral analysis and find the drag force D in terms of the flow properties ρ, U_0, and δ and the plate dimensions L and b.[10]

Solution

Like most practical cases, this problem requires a combined mass and momentum balance. A proper selection of control volume is essential, and we select the four-sided region from



0 to h to δ to L and back to the origin 0, as shown in Fig. 3.10. Had we chosen to cut across horizontally from left to right along the height $y = h$, we would have cut through the shear layer and exposed unknown shear stresses. Instead we follow the streamline passing through $(x, y) = (0, h)$, which is outside the shear layer and also has no mass flow across it. The four control volume sides are thus

1. From (0, 0) to (0, h): a one-dimensional inlet, $\mathbf{V} \cdot \mathbf{n} = -U_0$. $\vec{n}$：法向量.
2. From (0, h) to (L, δ): a streamline, no shear, $\mathbf{V} \cdot \mathbf{n} \equiv 0$.
3. From (L, δ) to (L, 0): a two-dimensional outlet, $\mathbf{V} \cdot \mathbf{n} = +u(y)$.
4. From (L, 0) to (0, 0): a streamline just above the plate surface, $\mathbf{V} \cdot \mathbf{n} = 0$, shear forces summing to the drag force $-D\mathbf{i}$ acting from the plate onto the retarded fluid.

The pressure is uniform, and so there is no net pressure force. Since the flow is assumed incompressible and steady, Eq. (3.37) applies with no unsteady term and fluxes only across sections 1 and 3:

$$\sum F_x = -D = \rho \int_1 \overset{\dot m_{in}}{\overset{\|}{u(0, y)(\mathbf{V} \cdot \mathbf{n})\, dA}} + \rho \int_3 \overset{\dot m_{out}}{\overset{\|}{u(L, y)(\mathbf{V} \cdot \mathbf{n})\, dA}}$$

$$= \rho \int_0^h U_0(-U_0) b\, dy + \rho \int_0^\delta u(L, y)[+u(L, y)] b\, dy$$

Evaluating the first integral and rearranging give

$$D = \rho U_0^2 b h - \rho b \int_0^\delta u^2 dy \,\Big|_{x=L} \qquad (1)$$

This could be considered the answer to the problem, but it is not useful because the height h is not known with respect to the shear layer thickness δ. This is found by applying mass conservation, since the control volume forms a streamtube:

$$\rho \int_{CS} (\mathbf{V} \cdot \mathbf{n})\, dA = 0 = \rho \int_0^h (-U_0) b\, dy + \rho \int_0^\delta u b\, dy \,\Big|_{x=L}$$

or

$$U_0 h = \int_0^\delta u\, dy \,\Big|_{x=L} \qquad (2)$$

after canceling b and ρ and evaluating the first integral. Introduce this value of h into Eq. (1) for a much cleaner result:

$$D = \rho b \int_0^\delta u(U_0 - u)\, dy \,\Big|_{x=L} \qquad \textit{Ans.}\ (3)$$

This result was first derived by Theodore von Kármán in 1921.[11] It relates the friction drag on one side of a flat plate to the integral of the *momentum deficit* $\rho u(U_0 - u)$ across the trailing cross section of the flow past the plate. Since $U_0 - u$ vanishes as y increases, the integral has a finite value. Equation (3) is an example of *momentum integral theory* for boundary layers, which is treated in Chap. 7.

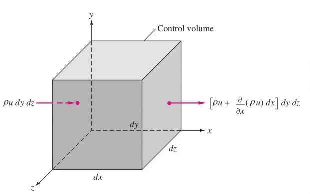
Chapter 4 微分.

1、The acceleration field of a fluid

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{Local}} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\text{Convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}$$

2. The differential Equation of Mass Conservation.

1). cartesian system (直角坐标)



mass conservation (chapter3): $\int_{cv} \frac{\partial \rho}{\partial t} dV + \dot{m}_{out} - \dot{m}_{in} = 0$

$\int_{cv} \frac{\partial \rho}{\partial t} dV \approx \frac{\partial \rho}{\partial t} dx\,dy\,dz$.

Face	Inlet mass flw	Outlet mass flw
x	$\rho u\, dy\, dz$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\, dx\right] dy\, dz$
y	$\rho v\, dx\, dz$	$\left[\rho v + \frac{\partial}{\partial y}(\rho v)\, dy\right] dx\, dz$
z	$\rho w\, dx\, dy$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w)\, dz\right] dx\, dy$

$\Rightarrow \frac{\partial \rho}{\partial t} dx\,dy\,dz + \frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz + \frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dz + \frac{\partial}{\partial z}(\rho w)\,dx\,dy\,dz = 0$

$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$

$\Leftrightarrow$ continuity relation: $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0$

2). cylindrical polar coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0$$

steady compressible flow: $\frac{\partial}{\partial t} \equiv 0$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0$$

incompressible flow: $\frac{\partial \rho}{\partial t} \approx 0$, $\rho = \text{const}$.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta) + \frac{\partial}{\partial z}(V_z) = 0$$

极坐标与直角坐标:

$\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r)$, $\frac{\partial v}{\partial y} = \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta)$, $\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(V_z)$

3. The differential equation of linear momentum

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

g：重力， T：粘性力.

$$\Leftrightarrow \rho\vec{g} - \vec{\nabla}P + \vec{\nabla}\cdot\vec{\tau}_{ij} = \rho\frac{d\vec{V}}{dt} = \rho\vec{a}$$

$$\left(\frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}\right)$$

☆ Navier-Stokes Equations:
newtonian fluid,
incompressible flow,
constant density and viscosity.

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\frac{du}{dt}$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\frac{dv}{dt}$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\frac{dw}{dt}$$

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EXAMPLE 4.5

Take the velocity field of Example 4.3, with $b = 0$ for algebraic convenience

$$u = a(x^2 - y^2) \qquad v = -2axy \qquad w = 0$$

and determine under what conditions it is a solution to the Navier-Stokes momentum equations (4.38). Assuming that these conditions are met, determine the resulting pressure distribution when z is "up" ($g_x = 0$, $g_y = 0$, $g_z = -g$).

Solution

- *Assumptions:* Constant density and viscosity, steady flow (u and v independent of time).
- *Approach:* Substitute the known (u, v, w) into Eqs. (4.38) and solve for the pressure gradients. If a unique pressure function $p(x, y, z)$ can then be found, the given solution is exact.
- *Solution step 1:* Substitute (u, v, w) into Eqs. (4.38) in sequence:

$$\rho(0) - \frac{\partial p}{\partial x} + \mu(2a - 2a + 0) = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = 2a^2\rho(x^3 + xy^2)$$

$$\rho(0) - \frac{\partial p}{\partial y} + \mu(0 + 0 + 0) = \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = 2a^2\rho(x^2y + y^3)$$

$$\rho(-g) - \frac{\partial p}{\partial z} + \mu(0 + 0 + 0) = \rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right) = 0$$

Rearrange and solve for the three pressure gradients:

$$\frac{\partial p}{\partial x} = -2a^2\rho(x^3 + xy^2) \qquad \frac{\partial p}{\partial y} = -2a^2\rho(x^2y + y^3) \qquad \frac{\partial p}{\partial z} = -\rho g \tag{1}$$

- *Comment 1:* The vertical pressure gradient is *hydrostatic*. (Could you have predicted this by noting in Eqs. (4.38) that $w = 0$?) However, the pressure is velocity-dependent in the xy plane.
- *Solution step 2:* To determine if the x and y gradients of pressure in Eq. (1) are compatible, evaluate the mixed derivative,$(\partial^2 p/\partial x\, \partial y)$; that is, cross-differentiate these two equations:

$$\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial y}[-2a^2\rho(x^3 + xy^2)] = -4a^2\rho xy$$

$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial x}[-2a^2\rho(x^2y + y^3)] = -4a^2\rho xy$$

- *Comment 2:* Since these are equal, the given velocity distribution is indeed an *exact* solution of the Navier-Stokes equations.
- *Solution step 3:* To find the pressure, integrate Eqs. (1), collect, and compare. Start with $\partial p/\partial x$. The procedure requires care! Integrate *partially* with respect to x, holding y and z constant:

$$p = \int \frac{\partial p}{\partial x}dx|_{y,z} = \int -2a^2\rho(x^3 + xy^2)\,dx|_{y,z} = -2a^2\rho\left(\frac{x^4}{4} + \frac{x^2y^2}{2}\right) + f_1(y, z) \tag{2}$$

Note that the "constant" of integration f_1 is a *function* of the variables that were not integrated. Now differentiate Eq. (2) with respect to y and compare with $\partial p/\partial y$ from Eq. (1):

$$\frac{\partial p}{\partial y}|_{(2)} = -2a^2\rho\, x^2y + \frac{\partial f_1}{\partial y} = \frac{\partial p}{\partial y}|_{(1)} = -2a^2\rho(x^2y + y^3)$$

$$\text{Compare: } \frac{\partial f_1}{\partial y} = -2a^2\rho\, y^3 \quad \text{or} \quad f_1 = \int \frac{\partial f_1}{\partial y}dy|_z = -2a^2\rho\frac{y^4}{4} + f_2(z)$$

$$\text{Collect terms: So far } \quad p = -2a^2\rho\left(\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{y^4}{4}\right) + f_2(z) \tag{3}$$

This time the "constant" of integration f_2 is a function of z only (the variable not integrated). Now differentiate Eq. (3) with respect to z and compare with $\partial p/\partial z$ from Eq. (1):

$$\frac{\partial p}{\partial z}|_{(3)} = \frac{df_2}{dz} = \frac{\partial p}{\partial z}|_{(1)} = -\rho g \qquad \text{or} \qquad f_2 = -\rho gz + C \tag{4}$$

where C is a constant. This completes our three integrations. Combine Eqs. (3) and (4) to obtain the full expression for the pressure distribution in this flow:

$$p(x, y, z) = -\rho gz - \tfrac{1}{2}a^2\rho(x^4 + y^4 + 2x^2y^2) + C \qquad \textit{Ans.} \tag{5}$$

This is the desired solution. Do you recognize it? Not unless you go back to the beginning and square the velocity components:

$$u^2 + v^2 + w^2 = V^2 = a^2(x^4 + y^4 + 2x^2y^2) \tag{6}$$

Comparing with Eq. (5), we can rewrite the pressure distribution as

$$p + \tfrac{1}{2}\rho V^2 + \rho gz = C \tag{7}$$

- *Comment:* This is Bernoulli's equation (3.54). That is no accident, because the velocity distribution given in this problem is one of a family of flows that are solutions to the Navier-Stokes equations and that satisfy Bernoulli's incompressible equation everywhere in the flow field. They are called *irrotational flows,* for which curl $\mathbf{V} = \nabla \times \mathbf{V} \equiv 0$. This subject is discussed again in Sec. 4.9.

4. The stream function. (incompressible, ρ-constant)

1). $\dot{m}_{in} = \dot{m}_{out} \Rightarrow$ 二维: $\rho u dy + \rho v dx = [\rho u + \frac{\partial}{\partial x}(\rho u)dx]dy + [\rho v + \frac{\partial}{\partial y}(\rho v)dy]dx \Rightarrow$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \equiv 0$$

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x} \Rightarrow \vec{V} = \vec{i}\frac{\partial \psi}{\partial y} - \vec{j}\frac{\partial \psi}{\partial x}$$

ψ = const along a streamline

$$dQ = (\vec{V}\cdot\vec{n})dA = \left(\vec{i}\frac{\partial \psi}{\partial y} - \vec{j}\frac{\partial \psi}{\partial x}\right)\cdot\left(\vec{i}\frac{dy}{ds} - \vec{j}\frac{dx}{ds}\right)ds$$

$$\Rightarrow Q_{1\to 2} = \int_1^2 (\vec{V}\cdot\vec{n})dA = \int_1^2 d\psi = \psi_2 - \psi_1$$

EXAMPLE 4.7

If a stream function exists for the velocity field of Example 4.5

$$u = a(x^2 - y^2) \quad v = -2axy \quad w = 0$$

find it, plot it, and interpret it.

Solution

- *Assumptions:* Incompressible, two-dimensional flow.
- *Approach:* Use the definition of stream function derivatives, Eqs. (4.85), to find $\psi(x, y)$.

- *Solution step 1:* Note that this velocity distribution was also examined in Example 4.3. It satisfies continuity, Eq. (4.83), but let's check that; otherwise ψ will not exist:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}[a(x^2 - y^2)] + \frac{\partial}{\partial y}(-2axy) = 2ax + (-2ax) \equiv 0 \quad \text{checks}$$

Thus we are certain that a stream function exists.

- *Solution step 2:* To find ψ, write out Eqs. (4.85) and integrate:

$$u = \frac{\partial \psi}{\partial y} = ax^2 - ay^2 \tag{1}$$

$$v = -\frac{\partial \psi}{\partial x} = -2axy \tag{2}$$

and work from either one toward the other. Integrate (1) partially

$$\psi = ax^2y - \frac{ay^3}{3} + f(x) \tag{3}$$

Differentiate (3) with respect to x and compare with (2)

$$\frac{\partial \psi}{\partial x} = 2axy + f'(x) = 2axy \tag{4}$$

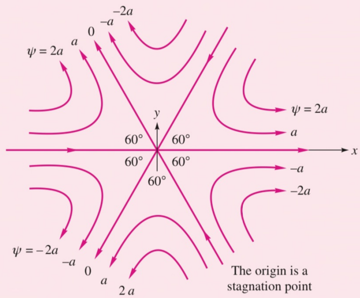
Therefore $f'(x) = 0$, or f = constant. The complete stream function is thus found:

$$\psi = a\left(x^2y - \frac{y^3}{3}\right) + C \qquad \text{解微分方程} \qquad \textit{Ans. } (5)$$

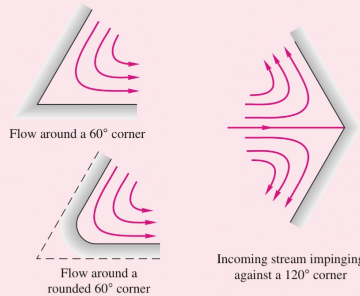
To plot this, set $C = 0$ for convenience and plot the function

$$3x^2y - y^3 = \frac{3\psi}{a} \tag{6}$$

for constant values of ψ. The result is shown in Fig. E4.7*a* to be six 60° wedges of circulating motion, each with identical flow patterns except for the arrows. Once the streamlines are labeled, the flow directions follow from the sign convention of Fig. 4.9. How



E4.7*a*



E4.7*b*

can the flow be interpreted? Since there is slip along all streamlines, no streamline can truly represent a solid surface in a viscous flow. However, the flow could represent the impingement of three incoming streams at 60, 180, and 300°. This would be a rather unrealistic yet exact solution to the Navier-Stokes equations, as we showed in Example 4.5.

By allowing the flow to slip as a frictionless approximation, we could let any given streamline be a body shape. Some examples are shown in Fig. E4.7*b*.

2). Steady plane compressible flow

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0,\quad \rho u=\frac{\partial \psi}{\partial y},\ \rho v=\frac{\partial \psi}{\partial x}$$

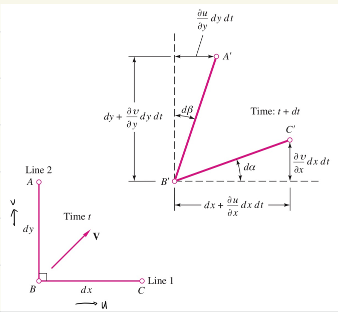
3). Incompressible plane flow in polar coordinates.

$V_z=0$:
$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta)=0,$$
$$V_r=\frac{1}{r}\frac{\partial \psi}{\partial \theta},\quad V_\theta=-\frac{\partial \psi}{\partial r},\quad V_z=0$$

轴对称

Axisymmetric Flow:
$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r)+\frac{\partial}{\partial z}(V_z)=0 \Rightarrow \frac{\partial}{\partial r}(rV_r)+\frac{\partial}{\partial z}(rV_z)=0,$$
$$V_r=-\frac{1}{r}\frac{\partial \psi}{\partial z},\quad V_z=\frac{1}{r}\frac{\partial \psi}{\partial r},\quad V_\theta=0$$

5. Vorticity and Irrotationally.



$$w_z \triangleq \frac{1}{2}\left(\frac{d\alpha}{dt}-\frac{d\beta}{dt}\right)$$ $w_z=0$ 无旋，$w_z\neq 0$ 有旋.

$$d\alpha=\lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial v/\partial x)dx\,dt}{dx+(\partial u/\partial x)dx\,dt}\right]=\frac{\partial v}{\partial x}dt$$

$$d\beta=\frac{\partial u}{\partial y}dt$$

$$\Rightarrow w_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right),\quad w_x=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\quad w_y=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right).$$

三维：$$w=\frac{1}{2}(\mathrm{curl}\,\vec{V})=\frac{1}{2}\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & w\end{vmatrix}.$$

黏性流体有旋，非黏性流体无旋.

potential lines : ϕ
$$u=\frac{\partial \phi}{\partial x},\quad v=\frac{\partial \phi}{\partial y},\quad w=\frac{\partial \phi}{\partial z}$$

极坐标：
$$\begin{cases} 2w_r=\frac{1}{r}\frac{\partial V_z}{\partial \theta}-\frac{\partial V_\theta}{\partial z}\\ 2w_\theta=\frac{\partial V_r}{\partial z}-\frac{\partial V_z}{\partial r}\\ 2w_z=\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)-\frac{1}{r}\frac{\partial V_r}{\partial \theta}.\end{cases}$$ ☆

$$V_z=0,\quad V_r=\frac{\partial \phi}{\partial r}=\frac{1}{r}\frac{\partial \psi}{\partial \theta},\quad V_\theta=\frac{1}{r}\frac{\partial \phi}{\partial \theta}=-\frac{\partial \psi}{\partial r}.$$

6. Orthogonality of Streamlines (ψ) and Potential Lines (ϕ)

正交性　　流函数　　势函数

$$\text{curl } \vec{V} = 0 \Rightarrow \begin{cases} \omega_x = 0 \\ \omega_y = 0 \\ \omega_z = 0 \end{cases} \Rightarrow \begin{cases} u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \\ w = \frac{\partial \phi}{\partial z} \end{cases}$$

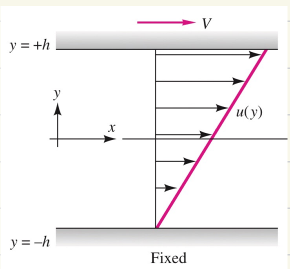
无旋

$$\begin{cases} d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0 = v\,dx - u\,dy \\ d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0 = u\,dx + v\,dy \end{cases} \Rightarrow \left(\frac{dy}{dx}\Big|_{\phi=c}\right)\cdot\left(\frac{dy}{dx}\Big|_{\psi=c}\right) = -1, \quad \psi \perp \phi$$

$$\begin{cases} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \Rightarrow \nabla^2 \psi = 0 \Rightarrow \phi \text{ exists.} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \Rightarrow \nabla^2 \phi = 0 \Rightarrow \psi \text{ exists} \end{cases}$$

极坐标：$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

7. Incompressible viscous flow between parallel plates



(a)

(a) no pressure gradient, upper plate moving. (Couette Flow)

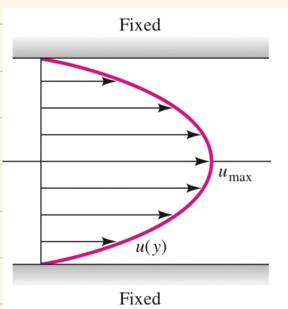
continuity equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial u}{\partial x} + 0 + 0 \Rightarrow u = u(y)$

N-S momentum equation: $\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

$$\rho(0+0) = 0 + 0 + \mu\left(0 + \frac{d^2 u}{dy^2}\right)$$

$$\Rightarrow u = C_1 y + C_2$$

$y = +h,\ u = V;\ y = -h,\ u = 0 \Rightarrow u = \frac{V}{2h}y + \frac{V}{2},\ -h \le y \le h$



(b)

(b) pressure gradient $\partial p/\partial x$ with both plate fixed.

continuity equation $\Rightarrow u = u(y)$

N-S momentum equation: $\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

$$\rho(0+0) = -\frac{\partial p}{\partial x} + 0 + \mu\left(0 + \frac{d^2 u}{dy^2}\right)$$

Since $u = u(y)$, $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \Rightarrow p = p(x)$

$$\mu\frac{d^2 u}{dy^2} = \frac{dp}{dx} = \text{const} < 0$$

$y = \pm h,\ u = 0 \Rightarrow u = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right)$

Chapter 5

1. Nondimensionalization of basic equations. 🔗P313

Continuity : $\vec{\nabla}\cdot\vec{V}=0$

Navier-Stokes : $\rho\frac{d\vec{V}}{dt}=\rho\vec{g}-\vec{\nabla}p+\mu\nabla^2\vec{V}$

$\Rightarrow$ Continuity : $\vec{\nabla}^*\cdot\vec{V}^*=0$

Momentum : $\frac{d\vec{V}^*}{dt^*}=-\vec{\nabla}^*p^*+\frac{\mu}{\rho UL}\nabla^{*2}(\vec{V}^*)$

Buckingham Pi Theorem

The procedure most commonly used to identify both the number and form of the appropriate non-dimensional parameters is referred to as the Buckingham Pi Theorem. The theorem uses the following definitions: $F=f(L,V,\rho,\mu)$

- n = the number of independent variables relevant to the problem $n=5$
- j' = the number of independent dimensions found in the n variables $j'=3$
- j = the reduction possible in the number of variables necessary to be considered simultaneously
- k = the number of independent **Π** terms that can be identified to describe the problem, k = n - j $k=2$，若不满足，$k=3,4,\ldots$

EXAMPLE 5.2

Repeat the development of Eq. (5.2) from Eq. (5.1), using the pi theorem.

Solution

Step 1 Write the function and count variables:

$$F = f(L, U, \rho, \mu) \quad \text{there are five variables } (n = 5)$$

Step 2 List dimensions of each variable. From Table 5.1

F	L	U	ρ	μ
$\{MLT^{-2}\}$	$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$

Step 3 Find j. No variable contains the dimension Θ, and so j is less than or equal to 3 (MLT). We inspect the list and see that L, U, and ρ cannot form a pi group because only ρ contains mass and only U contains time. Therefore j does equal 3, and $n - j = 5 - 3 = 2 = k$. The pi theorem guarantees for this problem that there will be exactly two independent dimensionless groups.

Step 4 Select repeating j variables. The group L, U, ρ we found in step 3 will do fine.

Step 5 Combine L, U, ρ with one additional variable, in sequence, to find the two pi products.

First add force to find Π_1. You may select *any* exponent on this additional term as you please, to place it in the numerator or denominator to any power. Since F is the output, or dependent, variable, we select it to appear to the first power in the numerator:

$$\Pi_1 = L^aU^b\rho^cF = (L)^a(LT^{-1})^b(ML^{-3})^c(MLT^{-2}) = M^0L^0T^0$$

Equate exponents:

Length: $a + b - 3c + 1 = 0$

Mass: $c + 1 = 0$

Time: $-b \quad -2 = 0$

We can solve explicitly for

$$a = -2 \qquad b = -2 \qquad c = -1$$

Therefore $\Pi_1 = L^{-2}U^{-2}\rho^{-1}F = \dfrac{F}{\rho U^2L^2} = C_F$ *Ans.*

This is exactly the right pi group as in Eq. (5.2). By varying the exponent on F, we could have found other equivalent groups such as $UL\rho^{1/2}/F^{1/2}$.

Finally, add viscosity to L, U, and ρ to find Π_2. Select any power you like for viscosity. By hindsight and custom, we select the power -1 to place it in the denominator:

$$\Pi_2 = L^aU^b\rho^c\mu^{-1} = L^a(LT^{-1})^b(ML^{-3})^c(ML^{-1}T^{-1})^{-1} = M^0L^0T^0$$

Equate exponents:

Length: $a + b - 3c + 1 = 0$

Mass: $c - 1 = 0$

Time: $-b \quad + 1 = 0$

from which we find

$$a = b = c = 1$$

Therefore $\Pi_2 = L^1U^1\rho^1\mu^{-1} = \dfrac{\rho UL}{\mu} = \text{Re}$ *Ans.*

Step 6 We know we are finished; this is the second and last pi group. The theorem guarantees that the functional relationship must be of the equivalent form

$$\frac{F}{\rho U^2L^2} = g\left(\frac{\rho UL}{\mu}\right) \qquad \textit{Ans.}$$

which is exactly Eq. (5.2).

Chapter 8

1. Circulation

$$\Gamma = \oint_c V\cos\alpha\, ds = \int_c \vec{V}\cdot d\vec{s} = \int_c (u\,dx + v\,dy + w\,dz) = \int d\phi$$

无旋流场 $\Gamma = 0$

$$\psi = -K\ln r \quad , \quad \phi = K\theta$$

2. Plane Flow Past Closed-Body Shapes.

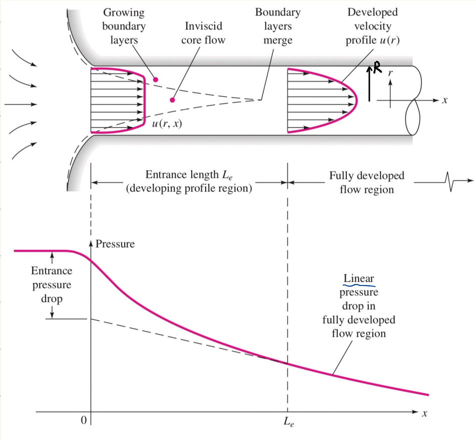
$$\psi = -\frac{\lambda\sin\theta}{r} \qquad \phi = \frac{\lambda\cos\theta}{r}.$$

$$\psi = U_\infty \sin\theta \left(r - \frac{a^2}{r}\right) - K\ln\frac{r}{a}$$

Ⅱ

Chapter 6

1. Fully developed laminar (层流) pipe flow (Chapter 4) 微分法



continuity : $\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta) + \frac{\partial}{\partial z}(V_z) = 0$

$$V_r = V_\theta = 0 \Rightarrow \frac{\partial}{\partial z}(V_z) = 0 \Rightarrow V_z = V_z(r)$$

r-momentum : $\frac{\partial V_r}{\partial t} + (\vec{V}\cdot\nabla)V_r - \frac{1}{r}V_\theta^2 = -\frac{1}{\rho}\frac{\partial P}{\partial r} + g_r + \nu\left(\nabla^2 V_r - \frac{V_r}{r^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}\right)$

$$V_r = V_\theta = g_r = 0 \Rightarrow \frac{\partial P}{\partial r} = 0 \Rightarrow P = P(z)$$

z-momentum : $\frac{\partial V_z}{\partial t} + (\vec{V}\cdot\nabla)V_z = -\frac{1}{\rho}\frac{\partial P}{\partial z} + g_z + \nu\nabla^2 V_z$

$$\Rightarrow \rho V_z\frac{\partial V_z}{\partial z} = -\frac{dP}{dz} + \mu\nabla^2 V_z = -\frac{dP}{dz} + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dV_z}{dr}\right)$$

$$\Rightarrow \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) = \frac{dP}{dt} = const < 0$$

$$\Rightarrow V_z = \frac{dP}{dz}\frac{r^2}{4\mu} + C_1\ln r + C_2$$

No slip at $r=R$: $V_z = 0 = \frac{dP}{dz}\frac{R^2}{4\mu} + C_1\ln R + C_2$.

Finite velocity at $r=0$: $V_z = \text{finite} = 0 + C_1\ln(0) + C_2$.

$$\boxed{V_z = \left(-\frac{dP}{dz}\right)\frac{1}{4\mu}(R^2 - r^2)}$$

$$\Rightarrow V_{max} = V_z(r=0) = \left(-\frac{dP}{dz}\right)\frac{R^2}{4\mu}$$

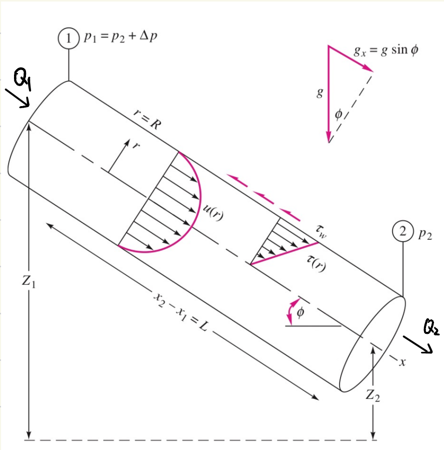
$$V_{avg} = \frac{1}{A}\int V_z dA = \frac{V_{max}}{2}.$$

$$Q = \int V_z dA = \frac{\pi R^4 \Delta P}{8\mu L}$$

$$\tau_{wall} = \frac{R}{2}\frac{\Delta P}{L}$$

2. Head loss – the friction factor

积分法.



$$Q_1 = Q_2 = const \Rightarrow V_1 = V_2 = V$$

energy equation : $\left(\frac{P}{\rho g} + \frac{\alpha V^2}{2g} + z\right)_1 = \left(\frac{P}{\rho g} + \frac{\alpha V^2}{2g} + z\right)_2 + h_f$

no pump or turbine, $\alpha_1 = \alpha_2 \Rightarrow h_f = (z_1 - z_2) + \left(\frac{P_1}{\rho g} - \frac{P_2}{\rho g}\right) = \Delta z + \frac{\Delta P}{\rho g}$.

momentum : $\sum F_x = \overbrace{\Delta P(\pi R^2)}^{press} + \overbrace{\rho g(\pi R^2)}^{gravity}\underbrace{L\sin\phi}_{\Delta z} - \overbrace{\tau_w(2\pi R)L}^{shear} = \dot{m}(V_2 - V_1) = 0$

$$\xRightarrow{\times\frac{1}{\rho g\pi R^2}} \Delta z + \frac{\Delta P}{\rho g} = h_f = \frac{2\tau_w L}{\rho g R} = \frac{4\tau_w}{\rho g}\frac{L}{d}.$$

$$h_f = f\frac{L}{d}\frac{V^2}{2g}, \text{ where } f = fcn(Re_d, \frac{\varepsilon}{d}, \text{duct shape}).$$

$$\Rightarrow f = \frac{8\tau_w}{\rho V^2}$$ 适用于任何流体.

$$Re = \frac{\rho V d}{\mu} = \frac{4\rho Q}{\pi\mu d}$$

3. Laminar fully developed pipe flow.

$$u = u_{max}\left(1 - \frac{r^2}{R^2}\right) \quad \text{where} \quad u_{max} = \left(-\frac{dp}{dx}\right)\frac{R^2}{4\mu} \quad \text{and} \quad \left(-\frac{dp}{dx}\right) = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)$$

$V_z = \ldots$

$$V = \frac{Q}{A} = \frac{u_{max}}{2} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{R^2}{8\mu}$$

$$Q = \int u\,dA = \pi R^2 V = \frac{\pi R^4}{8\mu}\left(\frac{\Delta p + \rho g \Delta z}{L}\right) \tag{6.12}$$

$$\tau_w = \left|\mu \frac{du}{dr}\right|_{r=R} = \frac{4\mu V}{R} = \frac{8\mu V}{d} = \frac{R}{2}\left(\frac{\Delta p + \rho g \Delta z}{L}\right)$$

$$h_f = \frac{32\mu L V}{\rho g d^2} = \frac{128\mu L Q}{\pi \rho g d^4}$$

$$f_{lam} = \frac{8\tau_{w,lam}}{\rho V^2} = \frac{64}{\rho V d/\mu} = \frac{64}{Re_d}.$$

*4. Turbulence Modeling

1). Reynolds' Time-Average Concept

fluctuating variables: $u = \bar{u} + u'$, $v = \bar{v} + v'$, $w = \bar{w} + w'$, $p = \bar{p} + p'$.

Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0$

Momentum: $\rho \frac{d\vec{V}}{dt} = -\vec{\nabla} p + \rho \vec{g} + \mu \nabla^2 \vec{V} \Rightarrow x:\ \rho \frac{d\bar{u}}{\partial t} = -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial}{\partial x}\left(\mu \frac{\partial \bar{u}}{\partial x} - \rho \overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial \bar{u}}{\partial z} - \rho \overline{u'w'}\right)$

$\Rightarrow \rho \frac{d\bar{u}}{dt} \approx -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial \tau}{\partial y}$ where $\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} = \tau_{lam} + \tau_{turb}$.

2). The Logarithmic Overlap Law

对数

overlap layer: $\frac{u}{u^*} = \frac{1}{k}\ln\frac{y u^*}{\nu} + B$, $u^* = \left(\frac{\tau_w}{\rho}\right)^{\frac{1}{2}}$ u: velocity, u^*: friction velocity.

$k \approx 0.41$, $B \approx 5.0$

*5. Turbulent Pipe Flow.

$\frac{u(r)}{u^*} \approx \frac{1}{k}\ln\frac{(R-r)u^*}{\nu} + B$, $u(r) = V = \frac{Q}{A} \Rightarrow \frac{V}{u^*} \approx 2.44\ln\frac{Ru^*}{\nu} + 1.34$

$u^* = \left(\frac{\tau_w}{\rho}\right)^{\frac{1}{2}}$, $f = \frac{8\tau_w}{\rho V^2} \Rightarrow \frac{V}{u^*} = \left(\frac{\rho V^2}{\tau_w}\right)^{\frac{1}{2}} = \left(\frac{8}{f}\right)^{\frac{1}{2}}$, $\frac{Ru^*}{\nu} = \frac{\frac{1}{2}Vd}{\nu}\cdot\frac{u^*}{V} = \frac{1}{2}Re_d\left(\frac{f}{8}\right)^{\frac{1}{2}}$

$\Rightarrow \frac{1}{f^{\frac{1}{2}}} \approx 1.99\log\left(Re_d f^{\frac{1}{2}}\right) - 1.02 \xrightarrow[\text{data better}]{\text{fit friction}} \frac{1}{f^{\frac{1}{2}}} = 2.0\log\left(Re_d f^{\frac{1}{2}}\right) - 0.8$

$$f = \begin{cases} 0.316\,\mathrm{Re}_d^{-1/4} & 4000 < \mathrm{Re}_d < 10^5 \quad \text{H. Blasius (1911)} \\ \left(1.8 \log \dfrac{\mathrm{Re}_d}{6.9}\right)^{-2} & \text{Ref. 9} \end{cases}$$

6. The Moody chart

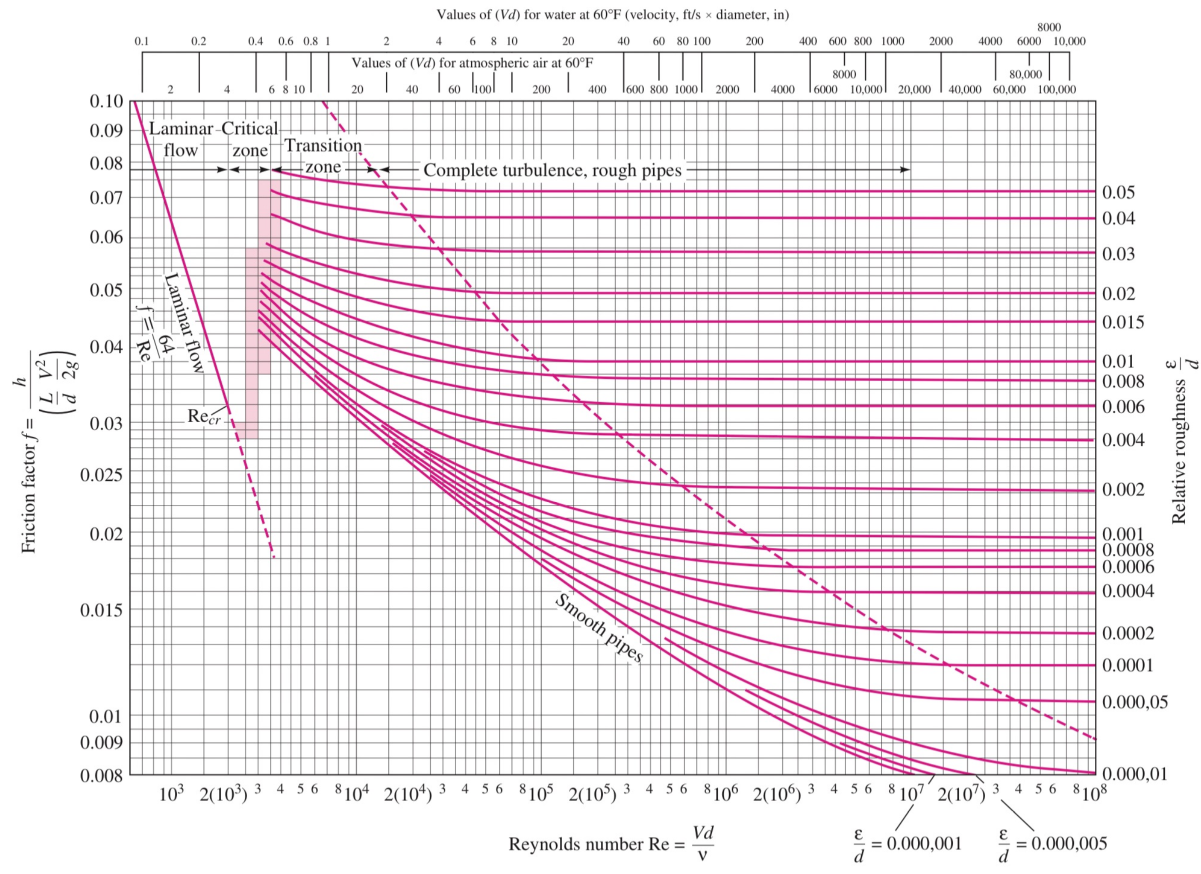


Fig. 6.13 The Moody chart for pipe friction with smooth and rough walls. This chart is identical to Eq. (6.48) for turbulent flow. *(From Ref. 8, by permission of the ASME.)*

Ex. 6.6, Ex 6.7

P371

7. Non-Circular Ducts.

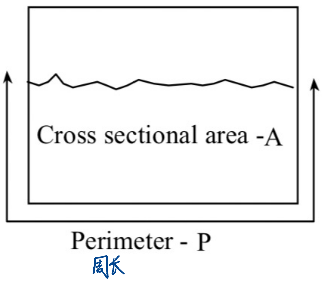
For flow in non-circular ducts or ducts for which the flow does not fill the entire cross-section, we can define the hydraulic diameter D_h as

$$D_h = \frac{4A}{P}$$

where

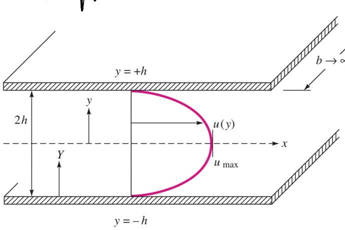
A = cross-sectional area of actual flow,

P = wetted perimeter, i.e. the perimeter on which viscous shear acts



With this definition, **<u>all previous equations</u>** for the Reynolds number, Re, friction factor, f, and head loss, h_f, are valid as previously defined and can be used on both circular and non-circular flow cross-sections.

Example 1

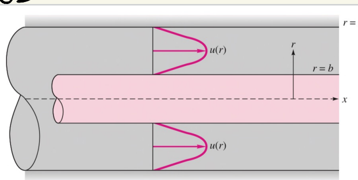


Probably the simplest noncircular duct flow is fully developed flow between parallel plates a distance $2h$ apart, as in Fig. 6.14. As noted in the figure, the width $b \gg h$, so the flow is essentially two-dimensional; that is, $u = u(y)$ only. The hydraulic diameter is

$$D_h = \frac{4A}{\mathcal{P}} = \lim_{b\to\infty} \frac{4(2bh)}{2b + 4h} = 4h \qquad (6.62)$$

that is, twice the distance between the plates. The pressure gradient is constant, $(-dp/dx) = \Delta p/L$, where L is the length of the channel along the x axis.

Example 2



The hydraulic diameter for an annulus is

$$D_h = \frac{4\pi(a^2 - b^2)}{2\pi(a + b)} = 2(a - b)$$

8. Minor or Local Losses in Pipe Systems.

ratio of head loss : $h_m = \Delta P/(\rho g)$, Loss coefficient : $K = \frac{h_m}{V^2/2g} = \frac{\Delta P}{\frac{1}{2}\rho V^2}$

$$\Delta h_{total} = h_f + \Sigma h_m = \frac{V^2}{2g}\left(\frac{fL}{d} + \Sigma K\right)$$

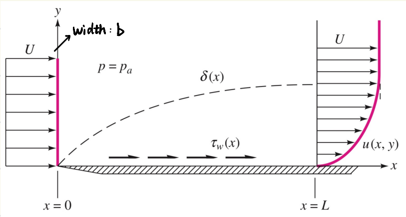
Chapter 7

1. Reynolds Number and Geometry Effects.

local Reynolds number : $Re_x = Ux/\nu$

$$\frac{\delta}{x} \approx \begin{cases} \dfrac{5.0}{Re_x^{1/2}} & \text{laminar} \quad 10^3 < Re_x < 10^6 \\ \dfrac{0.16}{Re_x^{1/7}} & \text{turbulent} \quad 10^6 < Re_x \end{cases}$$

2. Momentum Integral Estimates. P461～463



drag force on the plate : $D(x) = \rho b \int_0^{\delta(x)} u(U-u)\,dy$.

$D(x) = \rho b U^2 \theta$, $\theta \triangleq \int_0^{\delta} \frac{u}{U}\left(1-\frac{u}{U}\right) dy$

wall shear stress : $D(x) = b\int_0^x \tau_w(x)\,dx$

$\Rightarrow \frac{dD}{dx} = \rho b U^2 \frac{d\theta}{dx} = b\tau_w$

$\Rightarrow \tau_w = \rho U^2 \frac{d\theta}{dx}$ either laminar or turbulent.

laminar flow :

assume : $u = a_1 + a_2 y + a_3 y^2$, at $y=0$, $u=0$. at $y=\delta$, $\frac{du}{dy}=0$, $u=U$.

$\Rightarrow u(x,y) \approx U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)$, $0 \le y \le \delta(x)$

$\theta = \int_0^{\delta} \frac{u}{U}\left(1-\frac{u}{U}\right) dy = \int_0^{\delta} \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right) dy \approx \frac{2}{15}\delta$

$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} \approx \frac{2\mu U}{\delta}$

$\Rightarrow \delta\, d\delta \approx 15 \frac{\nu}{U} dx$, $\nu = \frac{\mu}{\rho}$

$\Rightarrow$ integrate from 0 to x : $\frac{1}{2}\delta^2 = \frac{15\nu x}{U}$

$\Rightarrow \frac{\delta}{x} \approx 5.5\left(\frac{\nu}{Ux}\right)^{\frac{1}{2}} = \frac{5.5}{Re_x^{1/2}}$

$\delta^* \approx \frac{1}{3}\delta$, $\frac{\delta^*}{x} \approx \frac{1.83}{Re_x^{1/2}}$

3. The Boundary Layer Equations. P465～466

Derivation for Two-Dimensional Flow: (incompressible, viscous, neglect gravity)

$$\begin{cases} \text{continuity} : \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ x\text{-momentum} : \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ y\text{-momentum} : \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{cases}$$

approximations: $\begin{cases} v \ll u \\ \dfrac{\partial u}{\partial x} \ll \dfrac{\partial u}{\partial y}, \quad \dfrac{\partial v}{\partial x} \ll \dfrac{\partial v}{\partial y} \\ Re_x = \dfrac{Ux}{\nu} \gg 1 \end{cases}$

$\Rightarrow \dfrac{\partial P}{\partial y} \approx 0 \Rightarrow P = P(x)$ only.

Bernoulli's equation: $\dfrac{dP}{\rho} + VdV + gdx = 0 \Longrightarrow \dfrac{\partial P}{\partial x} = \dfrac{dP}{dx} = -\rho U \dfrac{dU}{dx}$

(流体内外边界视作无粘)

Meanwhile: $\dfrac{\partial^2 u}{\partial x^2} \ll \dfrac{\partial^2 u}{\partial y^2}$.

$\Rightarrow \begin{cases} \text{continuity: } \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \text{momentum along wall: } u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} \approx U\dfrac{dU}{dx} + \dfrac{1}{\rho}\dfrac{\partial \tau}{\partial y} \end{cases}$ where $\tau = \begin{cases} \mu \dfrac{\partial u}{\partial y}, & \text{laminar} \\ \mu \dfrac{\partial u}{\partial y} - \rho \overline{u'v'}, & \text{turbulent.} \end{cases}$

With this result and the definition of the boundary layer thickness, the following key results are obtained for the laminar flat plate boundary layer:

Local boundary layer thickness $\qquad \delta(x) = \dfrac{5x}{\sqrt{Re_x}}$

Local skin friction coefficient:
(defined below) $\qquad C_{f_x} = \dfrac{0.664}{\sqrt{Re_x}}$

Total drag coefficient for length L (integration of τ_w dA over the length of the plate, per unit area, divided by $0.5\,\rho\, U_\infty^2$) $\qquad C_D = \dfrac{1.328}{\sqrt{Re_x}}$

where by definition $\boxed{C_{f_x} = \dfrac{\tau_w(x)}{\frac{1}{2}\rho U_\infty^2}}$ and $\boxed{C_D = \dfrac{F_D/A}{\frac{1}{2}\rho U_\infty^2}}$

A: 接触面积

With these results, we can determine local boundary layer thickness, local wall shear stress, and total drag force for laminar flow over a flat plate.

Laminar

$$\delta(x) = \frac{5x}{\sqrt{Re_x}}$$

$$C_{f_x} = \frac{0.664}{\sqrt{Re_x}}$$

$$C_D = \frac{1.328}{\sqrt{Re_L}}$$

Turbulent

$$\delta(x) = \frac{0.16\,x}{Re_x^{1/7}}$$

$$C_{f_x} = \frac{0.027}{Re_x^{1/7}}$$

$$C_D = \frac{0.031}{Re_L^{1/7}}$$

for turbulent flow over entire plate, 0 – L, i.e. assumes turbulent flow in the laminar region

4. The Flat-Plate Boundary Layer

Turbulent Flow.

$\tau_w(x) = \rho U^2 \frac{d\theta}{dx}$. $\Rightarrow C_f \triangleq \frac{\tau_w(x)}{\frac{1}{2}\rho U^2} = 2\frac{d\theta}{dx}$

$\frac{u}{u^*} \approx \frac{1}{\kappa}\ln\frac{y u^*}{\nu} + B$, $u^* = \left(\frac{\tau_w}{\rho}\right)^{\frac{1}{2}}$

outer edge of boundary layer : $y = \delta, u = U \Rightarrow \frac{U}{u^*} = \frac{1}{\kappa}\ln\frac{\delta u^*}{\nu} + B$

$\frac{U}{u^*} \equiv \left(\frac{2}{C_f}\right)^{1/2}$, $\frac{\delta u^*}{\nu} \equiv Re_\delta \left(\frac{C_f}{2}\right)^{1/2} \Rightarrow \left(\frac{2}{C_f}\right)^{1/2} \approx 2.44 \ln\left[Re_\delta \left(\frac{C_f}{2}\right)^{1/2}\right] + 5.0$ approximation : $C_f \approx 0.02 Re_\delta^{-\frac{1}{6}}$

Prandtl suggest : $\left(\frac{u}{U}\right)_{turb} \approx \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \Rightarrow \theta \triangleq \int_0^\delta \frac{u}{U}\cdot\left(1 - \frac{u}{U}\right)dy = \frac{7}{72}\delta$

$\Rightarrow C_f = 2\frac{d\theta}{dx} = 2\frac{d}{dx}\left(\frac{7}{72}\delta\right) = 0.02 Re_\delta^{-\frac{1}{6}} \Rightarrow Re_\delta^{-\frac{1}{6}} = 9.72\frac{d\delta}{dx} = 9.72\frac{d(Re_\delta)}{d(Re_x)}$

integrate : $Re_\delta \approx 0.16 Re_x^{6/7} \Rightarrow \frac{\delta}{x} \approx \frac{0.16}{Re_x^{1/7}}$, $C_f \approx \frac{0.027}{Re_x^{1/7}}$, $C_D = \frac{0.031}{Re_L^{1/7}}$.

5. Boundary Layers with Pressure Gradiant.

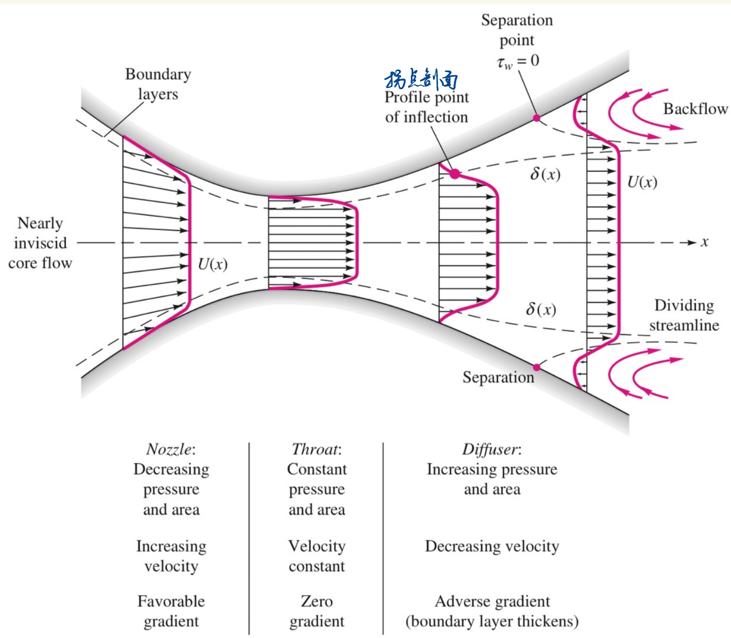


Fig. 7.8 Boundary layer growth and separation in a nozzle–diffuser configuration.

Chapter 9

1. The Perfect Gas 🔗P611～613

$p = \rho R T$, $R = C_p - C_v = \text{const}$, $k = \frac{C_p}{C_v} = \text{const}$

$R_{gas} = \frac{\Lambda}{M_{gas}}$, Λ, R, k 已知 (理想气体 $k=1.4$)

For air: $C_v = \frac{R}{k-1} = 718\ m^2/(s^2\cdot K)$, $C_p = \frac{kR}{k-1} = 1005\ m^2/(s^2\cdot K)$.

$\hat{u}_2 - \hat{u}_1 = C_v(T_2 - T_1)$, $h_2 - h_1 = C_p(T_2 - T_1)$

$\hat{u}$ (energy) $= \int C_v\, dT$, h (enthalpy 焓) $= \int C_p\, dT$

Isentropic Process 等熵过程

$T ds = dh - \frac{dp}{\rho}$, $dh = C_p dT$, $\rho T = p/R$

$\Rightarrow \int_1^2 ds = \int_1^2 C_p \frac{dT}{T} - R\int_1^2 \frac{dp}{p}$. $\Rightarrow s_2 - s_1 = C_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} = C_v \ln\frac{T_2}{T_1} - R\ln\frac{\rho_2}{\rho_1}$.

isentropic flow: $s_1 = s_2 \Rightarrow \frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} = \left(\frac{\rho_2}{\rho_1}\right)^k$.

☆ 2. The Speed of Sound. 🔗P614～616 (Fig 9.1)

the rate (传播速率) of propagation of a pressure pulse of infinitesimal strength through a still fluid.

continuity: $\rho A C = (\rho + \Delta\rho)(A)(C - \Delta V) \Rightarrow \Delta V = C\cdot\frac{\Delta\rho}{\rho+\Delta\rho}$

momentum: $\sum F = \dot{m}(V_{out} - V_{in}) \Rightarrow pA - (p+\Delta p)A = (\rho A C)(C - \Delta V - C) \Rightarrow \Delta p = \rho C \Delta V$

$$\Rightarrow C^2 = \frac{\Delta p}{\Delta \rho}\left(1 + \frac{\Delta\rho}{\rho}\right) \xrightarrow{\Delta\rho\to 0} a^2 \triangleq \frac{\partial p}{\partial \rho}. \Rightarrow a = \Big(\underbrace{\frac{\partial p}{\partial \rho}\Big|_s}_{\text{等熵}}\Big)^{1/2} = \Big(\underbrace{k\frac{\partial p}{\partial \rho}\Big|_T}_{\text{绝热}}\Big)^{1/2}$$

Perfect gas: $a = \left(\frac{kp}{\rho}\right)^{1/2} = (kRT)^{1/2}$.

3. Adiabatic (绝热的) and Isentropic Steady Flow 🔗P614～619 (Ex 9.3)

h_0: the stagnation enthalpy of the flow.

$h + \frac{1}{2}V^2 = h_0 = \text{const}$.

perfect gas: $\begin{cases} h = C_p T \Rightarrow C_p T + \frac{1}{2}V^2 = C_p T_0 \\ C_p T = \frac{kR}{k-1}T = \frac{a^2}{k-1} \end{cases}$, temperature absoulte zero: $V_{max} = (2h_0)^{1/2} = (2C_p T_0)^{1/2}$

$$\Rightarrow \left.\begin{aligned} &\frac{T_0}{T} = 1 + \frac{k-1}{2}Ma^2 , \quad Ma = \frac{V}{a} \\ &\frac{a_0}{a} = \left(\frac{T_0}{T}\right)^{1/2} = \left[1 + \tfrac{1}{2}(k-1)Ma^2\right]^{1/2} \end{aligned}\right\} \text{adiabatic flow}$$

(等熵一定绝热).

$$\left.\begin{aligned} &\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{k/(k-1)} = \left[1 + \tfrac{1}{2}(k-1)Ma^2\right]^{k/(k-1)} \\ &\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{1/(k-1)} = \left[1 + \tfrac{1}{2}(k-1)Ma^2\right]^{1/(k-1)} \end{aligned}\right\} \text{isentropic flow.}$$

(对于非等熵, p、ρ 为实际值, p_0、ρ_0 为等熵停滞的值)

4. Isentropic Flow with Area Changes. P622~623 (Fig 9.5)

continuity: $\rho(x)V(x)A(x) = \dot{m} = \text{const} \Rightarrow \frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$

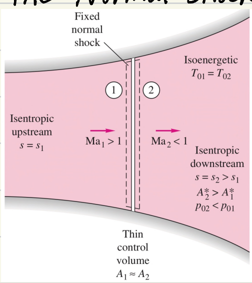
momentum: $\frac{dP}{\rho} + VdV = 0$

sound speed: $dp = a^2 d\rho$.

$$\Rightarrow \frac{d\rho}{\rho} = -\frac{VdV}{a^2} \Rightarrow -\frac{VdV}{a^2} + \frac{dV}{V} + \frac{dA}{A} = 0$$

$$\Rightarrow \frac{dV}{V} = \frac{dA}{A}\frac{1}{Ma^2-1} = -\frac{dP}{\rho V^2} \qquad \frac{A}{A^*} = \frac{1}{Ma}\frac{(1+0.2Ma^2)^3}{1.728}$$

5. The Normal Shock Wave.

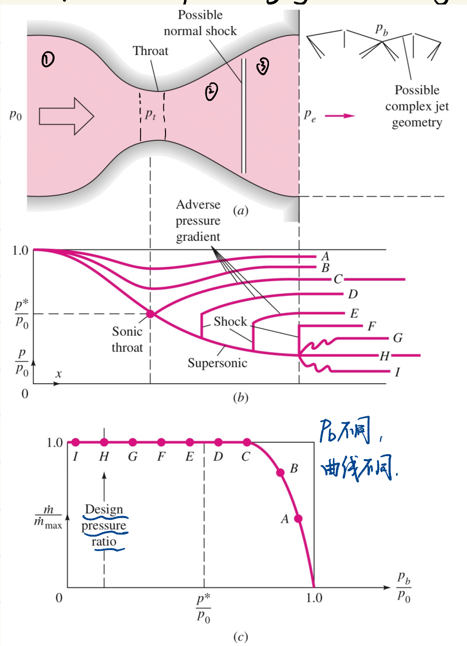


$$\frac{P_2}{P_1} = \frac{1}{k+1}[2k Ma_1^2 - (k-1)]$$

$$Ma_2^2 = \frac{(k-1)Ma_1^2 + 2}{2k Ma_1^2 - (k-1)}$$

Table B2

6. Operation of Converging and Diverging Nozzles.



P_b不同，曲线不同.

Table B1

A, B: $P_b = P_e$. subsonic through the nozzle, isentropic.

C: $P_b = P_e$. At throat (choked), sonic and Ma is unity.

①②③: subsonic and isentropic.

D, E: ① (Before throat): subsonic. TableB2

At throat, same as (C). A shock wave formed in ②~③

throat ~ ②: supersonic and accelerating.

③: subsonic and decelerating.

①, ②, ③: isentropic, shock: not isentropic.

F: At the exit, normal shock. Before shock, isentropic and supersonic; after shock, subsonic.

G: a series of two-dimensional shocks outside nozzle.

nozzle: isentropic

H: design pressure ratio. nozzle: isentropic.

I: same as (G), shocks: decelerate.

Shock: not isentropic and decelerate.